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LETTER TO THE EDITOR

The Virasoro constraint and Hirota's bilinear difference equation

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Abstract. From the perspective of Hirota's bilinear *difference* equation, the connection of the τ function and the conformal symmetry is shown to become apparent and the Virasoro constraint is naturally derived. In this formalism, the compatibility of the flow provided by conformal symmetry and the KP flow is manifest.

Nonlinear integrable systems and conformal field theory are closely related to each other in various senses. In particular in the matrix model approach to two-dimensional quantum gravity [1–3], a τ function of (reduced) KP hierarchy appears as the square root of a partition function, where the τ function solves certain bilinear differential equations, the so-called Hirota equations. In such matrix realizations of random surface models, a nonlinear ordinary differential equation can be derived for a specific heat, which describes multicritical points of a (p, q) minimal conformal field coupled to 2D gravity [4–6]. This equation is called the 'string equation'. For example, in the case $(p, q)=(3,2)$, which corresponds to pure gravity $(c=0)$, there appears a Painlevé equation of the first kind. The connection between the integrability of the Painlevé equations and that of the KP hierarchy is a very interesting subject to study.

From the perspective of the soliton equations, those differential equations in the matrix models can be viewed as extra conditions on the τ function besides the Hirota equations [7, 8]. Such conditions are known as $W^{(n)}$ conformal algebraic constraints. There appears again a quite non-trivial connection between the KP hierarchy and the conformal field theory. For example in one matrix model the corresponding algebra is the Z_2 twisted Virasoro algebra. In this letter we consider the case where one of the Virasoro constraints $(L_{-1}\tau=0)$ appears in connection with non-isospectral symmetries of the hierarchy, where the term 'non-isospectral' is due to the conformal reparametrization of the associated Riemann surface on which the τ function is defined. On the other hand, the ordinary KP flow of the τ function described by the Hirota equations corresponds to an isospectral flow. Intuitively speaking, the flow provided by the string equation is perpendicular to the KP flow.

It is known that the independent variables of the KP hierarchy, which are nothing but the KP time parameters, can be discretized without breaking the integrability of the original differential equations [9] in the sense that solutions of KdV equation simultaneously solve the (one-dimensional) Toda lattice equation [10]. The KdV equation and the (one-dimensional) Toda lattice equation are given by a reduction from the KP and Toda lattice hierarchy, respectively. Moreover, it can be shown that all of the variables in the KP and Toda lattice hierarchy are discretized in the above sense. So the KP and Toda lattice hierarchy can be rewritten as a difference or recursion equation. Such a universal difference equation was called Hirota's bilinear difference equation (HBDE). As can be seen from the above example, solution spaces of the KP hierarchy and the Toda lattice hierarchy have a non-trivial connection with each other. In fact the KP hierarchy is embedded in the Toda lattice hierarchy [12]. This means that the isospectral flow of the τ function guarantees the integrability in the case of a finite difference interval. This stability with respect to the discretization is a characteristic feature of completely integrable systems. It is interesting to know whether an analogous property exists or not in the flow provided by the string equation, i.e., is it stable or not under the discretization? Since it is known that a class of solutions to the Hirota equation of the KP hierarchy is given in terms of Riemann's theta function [13-15], this property of the solution of the Hirota equations can be considered as, from the point of view of the theta function, a result from Fay's triscecant formula for algebraic curves [16]. As an example, in the case of a genus-one Riemann surface the corresponding formula is the addition theorem of the Jacobi elliptic theta functions.

It was shown that the theta function solutions of HBDE, which are of course the τ function of the KP hierarchy simultaneously, can be interpreted as a correlation function of the conformal fields with arbitrary charges [17]. In this letter we show that an origin of the non-trivial connection of the τ function and the conformal algebras is an invariance of the τ function under the conformal transformation on the associated Riemann surface, if we consider the τ function as solutions to HBDE with the discretized variables. We essentially use the fact that conformal blocks satisfy the Knizhnik-Zamolodchikov equation [18]. We will find that the Virasoro constraints and the KP flow of the τ function are perpendicular each other and compatible.

Firstly, we review how HBDE can be constructed from the KP hierarchy and its solutions are given in terms of Riemann's theta function. Namely, we will show that the τ function of the KP hierarchy is identified with that of the HBDE. Then we see that it can be interpreted as the correlation function of a conformal field with arbitrary charges. The KP equation is the following partial differential equation,

$$\frac{\partial^2 u}{\partial y^2}=\frac{4}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}-\frac{1}{4}\frac{\partial^3 u}{\partial x^3}-3u\frac{\partial u}{\partial x}\right) \tag{1}$$

which has soliton-type analytic solutions. The KP hierarchy is a set of nonlinear differential equations like (1) with respect to infinite number of KP 'time' parameters $x \neq (x_1, x_2, x_3, \ldots)$. The independent variables in (1), i.e., x, y and t are nothing but x_1, x_2 and x_3, respectively. The KP equation (1) can be rewritten in bilinear form by replacing the dependent variable; $u(x, y, t) = (\partial^2/\partial x^2)\log f(x, y, t)$, then we obtain

$$(D_x^4+3D_y^2-4D_xD_t)f(x, y, t)f(x, y, t)=0 \tag{2}$$

where D_x is a Hirota derivative which operates to the bilinear form and is defined as

$$D_x f(x)g(x)=\frac{\partial}{\partial\eta}f(x+\eta)g(x-\eta)|_{\eta=0}.$$

It has been known that all of the equations in the KP hierarchy are reproduced from a single *difference* equation [9, 19], namely Hirota's bilinear difference equation (HBDE). To formulate the HBDE, we introduce a new set of independent variables $k_j (j=1, 2, \ldots)$, due to Miwa [19] instead of x_n. A variable transformation is written as

$$x_n=\frac{1}{2n}\sum_{j=1}^{\infty} z_j^{-n}k_j \tag{3}$$

where the z_j's are complex valued parameters in general. Then by using any three of these new variables the HBDE is written as follows:

$$\begin{aligned} z_1(z_2-z_3)f(k_1+1, k_2-1, k_3-1)f(k_1-1, k_2+1, k_3+1) \\ +z_2(z_3-z_1)f(k_1-1, k_2+1, k_3-1)f(k_1+1, k_2-1, k_3+1) \\ +z_3(z_1-z_2)f(k_1-1, k_2-1, k_3+1)f(k_1+1, k_2+1, k_3-1)=0. \end{aligned} \tag{4}$$

To see that the differential equations in the KP hierarchy are reconstructed from this equation, we take the infinitesimal limit of the parameters, z_j, in an appropriate order. In fact, if we consider the order of z_j's up to $O(z^6)$, e.g., $z_1^3z_2^2z_3$, we obtain the KP equation (2).

We show the τ function of the KP hierarchy solves the HBDE. It is well known that the Hirota equations are equivalent to the following bilinear identity for the 'wavefunction' [11]

$$\oint \frac{d\lambda}{2\pi i} w(x, \lambda)w^*(x', \lambda)=0 \tag{5}$$

where $w(x, \lambda)$ and $w^*(x', \lambda)$ are the wavefunction and its formal adjoint, respectively, and λ is a spectral parameter. We take an integration contour as a small circle around $\lambda=\infty$. In terms of the τ function the wavefunction and its adjoint are given by,

$$w(x, \lambda)=\frac{e^{\xi(x,\lambda)}\, e^{-\xi(\tilde\partial,\lambda^{-1})}\tau(x)}{\tau(x)} \tag{6a}$$

$$w^*(x', \lambda)=\frac{e^{-\xi(x',\lambda)}\, e^{\xi(\tilde\partial',\lambda^{-1})}\tau(x')}{\tau(x')} \tag{6b}$$

where $\xi(x, \lambda)=\sum_{n=1}^{\infty} x_n\lambda^n$, and $\tilde\partial =: (\partial/\partial x_1, \frac{1}{2}\partial/\partial x_2, \ldots)$. Conversely, we can derive the HBDE (4) starting from the bilinear identity (5). Let x and x' be rewritten as $x+y$ and $x-y$, and take the arbitrary parameters y_n as $y_n=-(1/2n)(z_1^{-n}+z_2^{-n}+z_3^{-n})$, then by using the fact

$$D_{k_j}=\sum_n \frac{1}{2n} z_j^{-n} D_{x_n}$$

we obtain (4) with $f=\tau$. Hence the τ function of KP hierarchy solves HBDE.

In terms of the variables introduced in (3), the theta function solutions of the KP hierarchy and hence those of HBDE is given by [13-15],

$$\tau(k, z)=\prod_{i,j=1}^{N}\left(\frac{E(z_i, z_j)}{z_i-z_j}\right)^{2k_ik_j}\vartheta\left(\zeta+\sum_{j=1}^{N} 2k_jw(z_j)\right). \tag{7}$$

In this solution $E(z_i, z_j)$ is a prime form, ζ is a constant vector and $w(z_j)=\int_0^{z_j}\omega_j$, where ω_j are the first Abel differentials associated with the Riemann surface, respectively. The remarkable fact is [17] that the τ function in this form is expressed as an N-point correlation function with 'charges' or momenta in string theory, k_j, at the jth external particle (figure 1). That is,

$$\begin{aligned} \tau(k, z)&=\left\langle 0\middle|:\prod_j V(k_j, z_j):G\middle|0\right\rangle \\ &=\frac{\langle 0|\Pi_j V(k_j, z_j)G|0\rangle}{\langle 0|\Pi_j V(k_j, z_j)|0\rangle} \end{aligned} \tag{8}$$

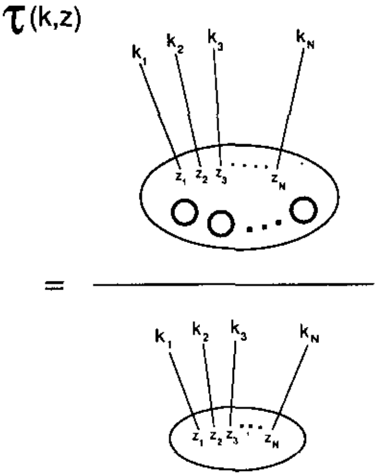

Figure 1. The τ function in terms of the dynamical variables k_j ('momenta') and the complex parameters z_j (positions of punctures) can be described by conformal blocks.

where $V(k_j, z_j)$ is the vertex operator constructed from a free boson and G is an operator which represents a loop configuration of Riemann surface whose explicit form is not needed here. The z_j are positions of punctures. In general, the k_js are subject to the condition $\Sigma_{j=1}^{N} k_j = 0$ which represents 'charge' or 'momentum' conservation in this picture, where N is the number of vertex operators.

We briefly comment on a reduction of the KP hierarchy. The l-reduced KP hierarchy [11] is given by imposing the extra condition on the τ function so that it does not depend on the 'time' x_n, where n are multiples of l, i.e. $(\partial/\partial x_{kl})\tau = 0$, $k = 1, 2, 3, \ldots$. In the correlation function interpretation of τ function, the l-reduction gives rise to the l-sheeted Riemann surface [20]. Namely, the corresponding conformal field is the twisted version of the one without reduction. Although in the matrix models the corresponding hierarchy is the l-reduced version, we consider only the KP hierarchy without reduction in this letter. The details of the l-reduced version and the connection to the matrix models will be presented elsewhere [21].

So far we see that the KP hierarchy is equivalent to the universal bilinear difference equation (HBDE). The theta function solutions of the HBDE can be interpreted as the correlation function of an arbitrary charge conformal fields on an arbitrary genus Riemann surface. At this point we recall that the correlation function (or string amplitude) was originally introduced as a function which is invariant under the conformal transformation, called dual symmetry in particle physics. From now on we show that the Virasoro constraint $L_{-1}\tau = 0$ is derived naturally by considering a reparametrization of the Riemann surfaces.

To clarify the 'direction' of the flow of the τ function, we rewrite (4) by using a difference operator,

$$K_H(z_j, D_{k_j}) = z_1(z_2 - z_3)\,\mathrm{e}^{D_{k_1} - D_{k_2} - D_{k_3}} + z_2(z_3 - z_1)\,\mathrm{e}^{-D_{k_1} + D_{k_2} - D_{k_3}} + z_3(z_1 - z_2)\,\mathrm{e}^{-D_{k_1} - D_{k_2} + D_{k_3}}. \qquad (9)$$

Then the HBDE is written in a compact form $K_H \tau(k, z)\tau(k, z) = 0$. It is apparent from this description that the KP (isospectral) flow is provided by taking k_j as dynamical variables, while perserving z_j constant.

We are now considering the variation of the τ function with z_j. The infinitesimal reparametrization of the Riemann surface is generally written as $z \to z' = z + \varepsilon_n z^{n+1}$. If we consider a variation of the τ function under the constant translation, i.e., $\varepsilon_n = 0$ except ε_{-1}, then we can explictly show that the theta function solutions of the HBDE (7) is invariant. That is

$$\delta\tau(k, z) = \varepsilon_{-1} \sum_j \frac{\partial}{\partial z_j} \tau(k, z) = 0. \tag{10}$$

Since by using the property of the prime form $E(z, z') = -E(z', z)$ and the fact that the first Abel differential is non-singular everywhere on Riemann surface, equation (10) is proved as,

$$\begin{aligned}
\sum_l \frac{\partial}{\partial z_j} \tau(k, z) \\
&= \sum_l \frac{\partial}{\partial z_l} \prod_{i,j=1}^{N} \left(\frac{E(z_i, z_j)}{z_i - z_j}\right)^{2k_ik_j} \vartheta\left(\zeta + \sum_{j=1}^{N} 2k_j w(z_j)\right) \\
&= \prod_{i,j=1}^{N} \left\{\sum_l \frac{\partial}{\partial z_l} (z_i - z_j)^{-2k_ik_j}\right\} E(z_i, z_j)^{2k_ik_j} \vartheta\left(\zeta + \sum_{j=1}^{N} 2k_j w(z_j)\right) \\
&\quad + \sum_{i,j=1}^{N} (z_i - z_j)^{-2k_ik_j} \left\{\sum_l \frac{\partial}{\partial z_l} E(z_i, z_j)^{2k_ik_j}\right\} \vartheta\left(\zeta + \sum_{j=1}^{N} 2k_j w(z_j)\right) \\
&\quad + \prod_{i,j=1}^{N} \left(\frac{E(z_i, z_j)}{z_i - z_j}\right)^{2k_ik_j} \left\{\sum_l \frac{\partial}{\partial z_l} \vartheta\left(\zeta + \sum_{j=1}^{N} 2k_j w(z_j)\right)\right\} \\
&= 0
\end{aligned} \tag{11}$$

where we have used the condition $\Sigma_{j=1}^{N} k_j = 0$. As will be seen later, (10) is nothing but the lowest order Virasoro constraint. Note that, from the correlation function interpretation of the τ function (equation (8)), we can also derive (10) by using the Knizhnik–Zamolodchikov equation of an Abelian version

$$\begin{aligned}
\kappa \frac{\partial}{\partial z_j} \langle V(k_1, z_1) V(k_2, z_2) \ldots V(k_j, z_j) \ldots \rangle \\
= \sum_i \frac{k_j k_i}{z_j - z_i} \langle V(k_1, z_1) V(k_2, z_2) \ldots V(k_j, z_j) \ldots \rangle
\end{aligned} \tag{12}$$

where κ is a numerical constant. Then we find from (8) and (10) that even on the multi-loop correlator (12) is always satisfied.

We now argue that the reparametrization $z \to z + \varepsilon_{-1}$ is equivalent to the Virasoro action L_{-1}, which preserves the values of discretized variables k_j. To see this we introduce the KP time variable and Fock representation of the Virasoro algebra. Firstly we define the Heisenberg subalgebra,

$$[a_k, a_l] = k\delta_{k-l} \qquad k, j \in \mathbb{Z}. \tag{13}$$

This algebra is realized by using the KP time x_n and its differential operator $\partial/\partial x_n$, such that,

$$a_k = \begin{cases} -kx_{-k} & k<0 \\ 0 & k=0 \\ \partial/\partial x_k & k>0. \end{cases} \tag{14}$$

In terms of these operators we can define the Fock representation of the Virasoro algebra,

$$L_n = \tfrac{1}{2} \sum_{j=-\infty}^{\infty} :a_j a_{n-j}: \tag{15}$$

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{16}$$

We do not now consider central extension. When the dynamical variables k_j are fixed constant in (3), the KP time changes under the constant translation $z \to z' = z + \varepsilon_{-1}$ as

$$x_m \to x'_m = x_m - (m+1)\varepsilon_{-1} x_{m+1}. \tag{17}$$

Explicitly, we can write,

$$\begin{aligned} x_1 &\to x'_1 = x_1 - 2\varepsilon_{-1}x_2 \\ x_2 &\to x'_2 = x_2 - 3\varepsilon_{-1}x_3 \\ x_3 &\to x'_3 = x_3 - 4\varepsilon_{-1}x_4 \end{aligned} \tag{18}$$

and so on. Then the variation of the τ function is found to be

$$\begin{aligned} \tau(x_1, x_2, \ldots) &\to \tau(x_1 - 2\varepsilon_{-1}x_2, x_2 - 3\varepsilon_{-1}x_3, \ldots) \\ &= \tau(x_1, x_2, \ldots) - \varepsilon_{-1}\left(2x_2 \frac{\partial}{\partial x_1} + 3x_3 \frac{\partial}{\partial x_2} + \ldots\right)\tau(x_1, x_2, \ldots) \\ &= \tau(x_1, x_2, \ldots) - \varepsilon_{-1} \sum_{j\geqslant 1} (j+1)x_{j+1} \frac{\partial}{\partial x_j} \tau(x_1, x_2, \ldots). \end{aligned} \tag{19}$$

The generator of the translation of the τ function in this form is nothing but L_{-1} from (15).

In the matrix models, as was mentioned previously, there appears the τ functions of the l-reduced version of the KP hierarchy in which the corresponding conformal algebras are the twisted version. It was shown [8, 22] that, in the 2-reduced case, i.e., one matrix model, the Virasoro constraints $L_n\tau = 0$, $(n \geqslant 0)$ are derived recursively from the string equation, which is equivalent to the lowest order constraint $L_{-1}\tau = 0$, in addition to the KdV (2-reduced KP) flow equation on the τ function. Similarly, as conjectured in [7], it is expected that the W constraints can also be derived from $L_{-1\tau} = 0$ and the l-reduced KP flow. In our case, however, there is no proof the same situation will occur. We have shown that the constraint $L_{-1}\tau = 0$; then by using the KP flow equation, we can obtain the other constraints needed to determine the τ function completely.

In summary, as can be seen from this result, we find that the class of the τ function given by the theta function is always subject to the condition (10) in addition to the HBDE which provides the KP (isospectral) flow. In particular, in the language of differential equations (Hirota equations), the Virasoro condition $L_{-1}\tau = 0$ is naturally derived by using the KP time. It is found that in terms of KP time parameters, the KP (isospectral) flow and non-isospectral symmetry are mixed in the action on the τ function, while by using 'momenta' k_j as the dynamical variables they are completely decoupled. The compatibility of both flows is apparent in terms of the discretized variables in HBDE.

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